Copolymers at selective interfaces: settled issues and open problems

Francesco Caravenna, Giambattista Giacomin and Fabio Lucio Toninelli

Abstract We review the literature on the localization transition for the class of polymers with random potentials that goes under the name of *copolymers near selective interfaces*. We outline the results, sketch some of the proofs and point out the open problems in the field. We also present in detail some alternative proofs that simplify what one can find in the literature.

Key words: Directed Polymers, Disorder, Localization, Copolymers at Selective Interfaces, Rare-Stretch Strategies, Fractional Moment Estimates.

2010 Mathematics Subject Classification: 60K35, 82B41, 82B44.

1 Copolymers and selective solvents

1.1 A basic model

In [13], T. Garel, D. A. Huse, S. Leibler and H. Orland introduced a simple model in order to look into how *the statistical behavior of macromolecules can be strongly affected by randomness in the physico-chemical properties of their constituents.* They

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aimed at a special class of macromolecules of linear type, the *random hydrophilic-hydrophobic copolymers*, in a medium of water and oil, separated by an interface. Such a polymer chain is just made up of monomers that differ for their affinity for water (or oil). The affinity is reduced to a real parameter that we call *charge*: the charge of the *j*-th monomer is denoted in [13] by ζ_j and, in mathematical terms, $\{\zeta_j\}_{j=1,2,\dots}$ is an IID sequence of Gaussian random variables, with given mean and variance. In order both to conform with the mathematical literature and to generalize slightly the problem we will write ζ_j as $\omega_j + h$, where $h \in \mathbb{R}$ (or $h \geq 0$ as we will do next) and $\omega = \{\omega_j\}_{j=1,2,\dots}$ is an IID sequence of random variables (often referred to as the *disorder*) such that

$$\mathrm{M}(t) := \mathbb{E}\left[\exp\left(t\omega_1\right)\right] < \infty, \tag{1}$$

for every $t \in \mathbb{R}$ and such that $\mathbb{E}\omega_1 = 0$, $\mathbb{E}\omega_1^2 = 1$. Apart for the larger class of charges that we allow and for notations, the Hamiltonian of the polymer model set forth in [13] is

$$H_{N,\omega,h}(S) := \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}(S_n), \tag{2}$$

where N is the length of the polymer and $S = \{S_0, S_1, \dots\}$ is a simple symmetric random walk trajectory ($S_0 = 0$, $\{S_{n+1} - S_n\}_{n=0,1,\dots}$ is a sequence of independent identically distributed symmetric random variables that take only values ± 1: the law of S is denoted by $\mathbf{P}$ and we stress that ω and S are independent).

We invite the reader to have a look from now at Figure 1 for the directed polymer interpretation of the trajectories of the model. A small detail to deal with is $\mathrm{sign}(0)$: $\mathrm{sign}(S_n)$ should be read as $\mathrm{sign}(S_{n-1})$ when $S_n = 0$ and this convention is particularly natural in directed polymer terms, because $\mathrm{sign}(S_n)$ is $+1$ (-1) if the nth monomer is in the upper (lower) half plane, that is in oil (water), see Figure 1. Still to conform with most of the mathematical literature on copolymers, the inverse temperature is denoted by $\lambda (\geq 0)$ instead of the more customary β, so that the Boltzmann factor that defines the polymer model of length N is $\exp(\lambda H_{N,\omega,h}(S))$. We are interested in the *quenched* system so we underline the very different nature of the two sources of randomness: ω is chosen once for all at the beginning of the experiment (the hydrophilic or hydrophobic character of the monomers does not change, while the chain fluctuates).

At a superficial level the effect of the charges on the polymer is quite intuitive: for $\lambda > 0$ positively charged monomers ($\omega_n + h > 0$, that is hydrophobic monomers) prefer lying in the upper half-plane (oil) and the opposite is true for the negatively charged ones. But for large N these *energetically favorable* trajectories become more and more atypical for $\mathbf{P}$ since placing the monomers in their preferred solvent strongly reduces the fluctuation freedom of the chain. We are therefore dealing with an energy-entropy competition that in the limit $N \to \infty$ leads to a localization-delocalization transition: localization arises when energy prevails and the polymer sticks to the oil-water interface, visiting thus both oil and water, while delocalization corresponds to the case in which the polymer prefers to stay away from the interface. We will come back to this with much more details, but we an-
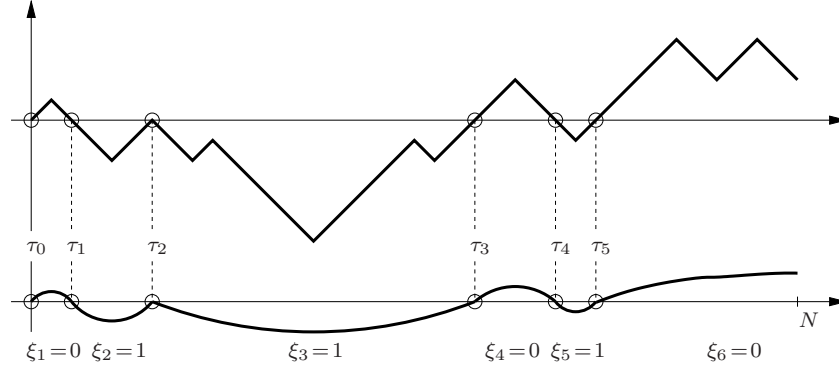
Fig. 1 The upper part of the figure shows a copolymer configuration: each *bond*, or segment, $[(n-1, S_{n-1}), (n, S_n)]$ of the random walk trajectory represents a monomer, so that sign(S_n) should be read as $+1$ (resp. -1) if this monomer is in the upper (resp. lower) half-plane. Note that the Hamiltonian of the copolymer, cf. (2), does not depend on the details of S within an excursion, but only on the length and sign of the excursion. This naturally leads to the generalized model introduced in section 1.2 (lower part of the figure) in terms of a general discrete renewal process τ and a sequence of independent signs ξ. It is mathematically useful to use the variable Δ_n, that in the random walk case is just $(1 - \text{sign}(S_n))/2$, *i.e.* the indicator function that the copolymer is below the axis. Note in fact that, by (2), $H_{N,\omega,h}(S) = -2\sum_{n=1}^{N}(\omega_n + h)\Delta_n + c_N(\omega)$, with $c_N(\omega) = \sum_{n=1}^{N}(\omega_n + h)$ which does not depend on S (or Δ), therefore we can drop it without changing the polymer measure (that is what we do in (4)).

ticipate that this entropy-energy competition turns out to be rather challenging: the phase diagram of the model is for the moment only partially understood and sound conjectures (or even only convincing physical heuristics) are lacking on several fundamental issues. This may appear rather surprising in view of the very simple nature of the model and of the fact, at the heart of the motivation of [13], that it represents one of the simplest instances of a general mechanism that plays a crucial role in a variety of extremely important phenomena (protein folding, to name one).

1.2 The (general) copolymer model

As argued in Figure 1 and its caption, the basic copolymer model does not depend on all the details of S, but just on its zero level set, which is a renewal set, and on the sign of the excursions, that is simply an independent fair coin tossing sequence. It is therefore natural, and at times really helpful, to look at the following generalized framework. Let us consider a discrete renewal process $\tau = \{\tau_n\}_{n \geq 0}$ on the non-negative integers $\mathbb{N} \cup \{0\}$, *i.e.*, a sequence of random variables such that $\tau_0 = 0$ and $\{\tau_{j+1} - \tau_j\}_{j=0,1,\dots}$ is an IID sequence of positive integer-valued variables, with marginal law satisfying

$$K(n) := \mathbf{P}(\tau_1 = n) \overset{n \to \infty}{\sim} \frac{c_K}{n^{1+\alpha}}, \tag{3}$$

where $\alpha > 0$ and $c_K > 0$ (we write $f_n \sim g_n$ for $f_n/g_n \to 1$). Since we are dealing with renewal processes it is important to stress that $\mathbf{P}(\tau_1 < \infty) = 1$, that is $\sum_{n=1}^{\infty} K(n) = 1$, so that τ is a *persistent* renewal. More generally, one could replace the constant c_k in (3) by a slowly varying function $L(n)$, but we stick for simplicity to the purely polynomial asymptotic behavior (3). It is well-known that the first return time to zero of the simple symmetric random walk $\inf\{n > 0 : S_n = 0\}$ satisfies (3) (restricted to the even integers, due to the usual periodicity issue) with $\alpha = \frac{1}{2}$. In particular, the basic model presented in the previous subsection is a special case of the generalized copolymer model we are defining, as it will be clear in a moment.

Remark 1. It is practical to switch freely from looking at τ as a sequence to considering it a *random set*, so for example $|\tau \cap [0,N]|$ is the number of renewals up to N, or $n \in \tau$ is the event that there exists j such that $\tau_j = n$. For a comprehensive references on renewal processes see for example [2].

The renewal τ identifies the polymer-interface contacts: we still need to know whether the excursion is above or below the axis. For this let $\xi = \{\xi_n\}_{n \in \mathbb{N}}$ denote an IID sequence of $B(1/2)$ variables (that is $\mathbf{P}(\xi_n = 0) = \mathbf{P}(\xi_n = 1) = \frac{1}{2}$) independent of τ, that we still call signs. Starting from the couple (τ, ξ) we build a new sequence $\Delta = \{\Delta_n\}_{n \in \mathbb{N}}$ by setting $\Delta_n = \sum_{j=1}^{\infty} \xi_j \mathbf{1}_{(\tau_{j-1}, \tau_j]}(n)$, in analogy with the simple random walk case: the signs Δ_n are constant between the points in τ and they are determined by ξ.

By *copolymer model* we mean the probability law $\mathbf{P}_{N,\omega} = \mathbf{P}_{N,\omega,\lambda,h}$ for the sequence Δ defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\Delta) := \frac{1}{Z_{N,\omega}} \exp\left(-2\lambda \sum_{n=1}^{N} \Delta_n(\omega_n + h)\right), \tag{4}$$

where $N \in \mathbb{N}$, $\lambda \geq 0$, $h \in \mathbb{R}$ (but we can and will assume $h \geq 0$ without loss of generality) and $\omega = \{\omega_n\}_{n \in \mathbb{N}}$ has been introduced in § 1. The partition function $Z_{N,\omega} = Z_{N,\omega,\lambda,h}$ is given by

$$Z_{N,\omega} := \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} \Delta_n(\omega_n + h)\right)\right]. \tag{5}$$

In order to emphasize the value of α in (3), we will sometimes speak of a α-*copolymer model*, but $\mathbf{P}_{N,\omega}$ depends on the full distribution $K(\cdot)$, not only on α.

1.3 The free energy: localization and delocalization

We introduce the free energy of the copolymer by

$$\mathrm{F}(\lambda,h) := \lim_{N\to\infty} \mathrm{F}_N(\lambda,h), \qquad \text{where} \qquad \mathrm{F}_N(\lambda,h) := \frac{1}{N}\mathbb{E}\left[\log Z_{N,\omega,\lambda,h}\right]. \quad (6)$$

The existence of such a limit follows by a standard argument based on super-additivity, see for example [14] or [15, Ch. 4], where it is also proven that

$$\mathrm{F}(\lambda,h) = \lim_{N\to\infty} \frac{1}{N}\log Z_{N,\omega,\lambda,h}, \qquad \mathbb{P}(\mathrm{d}\omega)\text{-a.s. and in } L^1(\mathbb{P}). \quad (7)$$

Equations (6)–(7) are telling us that the limit in (7) does not depend on the (typical) realization of ω, however it does depend on $\mathbb{P}$, that is on the law of ω_1, as well as on the inter-arrival law $K(\cdot)$. This should be kept in mind, even if we omit $\mathbb{P}$ and $K(\cdot)$ from the notation $\mathrm{F}(\lambda,h)$. Let us point out from now that $\mathrm{F}(\lambda,\cdot)$ and $\mathrm{F}(\cdot,h)$ are convex functions, since they are limits of convex functions. As a matter of fact $\mathrm{F}(\cdot,\cdot)$ is only separately convex because of the choice of the parametrization, but it is straightforward to see that $(\lambda,h) \mapsto \mathrm{F}(\lambda,h/\lambda)$ is convex.

Remark 2. It is sometimes useful to consider the *constrained* partition function $Z_{N,\omega}^{\mathsf{c}} = Z_{N,\omega,\lambda,h}^{\mathsf{c}}$ of the model, defined by

$$Z_{N,\omega,\lambda,h}^{\mathsf{c}} := \mathbf{E}\left[\exp\left(-2\lambda\sum_{n=1}^{N}(\omega_n + h)\Delta_n\right)\mathbf{1}_{\{N\in\tau\}}\right], \quad (8)$$

which differs from (5) only by the *boundary condition* factor $\mathbf{1}_{\{N\in\tau\}}$. It is a standard fact [15, Remark 1.2] that for all N,λ,h we have

$$Z_{N,\omega,\lambda,h}^{\mathsf{c}} \le Z_{N,\omega,\lambda,h} \le CN Z_{N,\omega,\lambda,h}^{\mathsf{c}}, \quad (9)$$

where C is a positive constant. In particular, the free energy $\mathrm{F}(\lambda,h)$ does not change if $Z_{N,\omega,\lambda,h}$ is replaced by $Z_{N,\omega,\lambda,h}^{\mathsf{c}}$ in (6) and (7). Furthermore, since $N \mapsto \mathbb{E}(\log Z_{N,\omega,\lambda,h}^{\mathsf{c}})$ is a real super-additive sequence, we can write

$$\mathrm{F}(\lambda,h) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}(\log Z_{N,\omega,\lambda,h}^{\mathsf{c}}) = \sup_{N\in\mathbb{N}} \frac{1}{N}\mathbb{E}(\log Z_{N,\omega,\lambda,h}^{\mathsf{c}}). \quad (10)$$

A crucial observation is:

$$\mathrm{F}(\lambda,h) \ge 0 \text{ for every } \lambda,h \ge 0. \quad (11)$$

This follows by restricting the expectation in (5) to the event $\{\tau_1 > N, \xi_1 = 0\}$, on which we have $\Delta_1 = 0, \ldots, \Delta_N = 0$, hence we obtain $Z_{N,\omega} \ge \frac{1}{2}\mathbf{P}(\tau_1 > N)$ and it suffices to observe that $N^{-1}\log\mathbf{P}(\tau_1 > N)$ vanishes as $N \to \infty$, thanks to (3). Notice that the event $\{\tau_1 > N, \xi_1 = 0\}$ corresponds to the set of trajectories that never visit the lower half plane, therefore the right hand side of (11) may be viewed as the contribution to the free energy given by these trajectories.

Based on this, we say that $(\lambda,h) \in \mathscr{D}$ (*delocalized regime*) if $\mathrm{F}(\lambda,h) = 0$, while $(\lambda,h) \in \mathscr{L}$ (*localized regime*) if $\mathrm{F}(\lambda,h) > 0$. This may look at first as a cheap way

to escape from the real localization/delocalization issue, that is inherently linked to the path properties of the measure $\mathbf{P}_{N,\omega}$, but it is not the case. Notice in fact that, if $h \mapsto \mathrm{F}(\lambda, h)$ is differentiable (which fails at most for a countable number of values of h, by convexity), by differentiating (7) and by convexity arguments we have

$$-\frac{1}{2\lambda}\frac{\partial}{\partial h}\mathrm{F}(\lambda, h) = \lim_{N\to\infty}\mathbf{E}_{N,\omega}\left[\frac{\mathscr{N}_N}{N}\right], \qquad \mathbb{P}\text{-a.s.,} \tag{12}$$

where

$$\mathscr{N}_N := \sum_{n=1}^{N}\Delta_n, \tag{13}$$

is just the total number of monomers in the lower half-plane, that is in water (cf. Figure 1). Therefore if (λ, h) is chosen in the interior of $\mathscr{D}$, where $\mathrm{F} \equiv 0$, the polymer visits water with null density ($\mathscr{N}_N/N \to 0$). On the other hand, if (λ, h) is in $\mathscr{L}$, the polymer puts a positive density, precisely $-(1/(2\lambda))\partial\mathrm{F}(\lambda, h)/\partial h \in (0, 1)$, of monomers in water and the rest, still a positive density, in oil.

We will deal below, *cf.* section 5, with sharper results on path behavior, but the elementary observation we have just made shows that the definition we have set forth of localization and delocalization is far from being artificial. As a matter of fact, it is the natural physical definition, and in fact it has been used already in [13], while in the mathematical literature was first introduced by [7].

1.4 The phase diagram

Convexity and the evident monotonicity of $\mathrm{F}(\lambda, \cdot)$ put strong a priori constraints on the phase diagram: let us go through this before going toward sharper questions. We can set

$$h_c(\lambda) := \sup\{h : \mathrm{F}(\lambda, h) > 0\}, \tag{14}$$

and the monotonicity of $\mathrm{F}(\lambda, \cdot)$ guarantees that $(\lambda, h) \in \mathscr{L}$ if and only if $h < h_c(\lambda)$, namely that $h_c(\cdot)$ is the critical curve. Let us derive a number of elementary properties of $h_c(\cdot)$.

The fact that $h_c(\lambda) < \infty$ for every λ follows by the standard *annealed bound*:

$$\mathbb{E}\log Z_{N,\omega} \leq \log\mathbb{E}Z_{N,\omega} = \log\mathbf{E}\mathbb{E}\left[\exp\left(-2\lambda\sum_{n=1}^{N}\Delta_n(\omega_n + h)\right)\right] =$$

$$\log\mathbf{E}\exp\left((\log\mathrm{M}(-2\lambda) - 2\lambda h)\sum_{n=1}^{N}\Delta_n\right), \quad (15)$$

so that $\mathrm{F}(\lambda, h) \leq 0$ (hence, recall (11), $\mathrm{F}(\lambda, h) = 0$) if $h \geq \log\mathrm{M}(-2\lambda)/(2\lambda)$, and $\log\mathrm{M}(-2\lambda)/(2\lambda) < \infty$ for every λ by (1).

Remark 3. The exponential of the rightmost term in (15) is the partition function of the *annealed* model associated to our quenched model. The free energy of the annealed model is rather trivial: it is in fact an elementary exercise to see that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbf{E} \exp\left((\log M(-2\lambda) - 2\lambda h) \sum_{n=1}^{N} \Delta_n \right) = (\log M(-2\lambda) - 2\lambda h)_+, \quad (16)$$

where $a_+ := a\mathbf{1}_{a>0}$. The annealed model has therefore a (de)localization transition too: its critical curve is $h_c^{ann}(\lambda) := \log M(-2\lambda)/(2\lambda)$ and we have just remarked that $h_c(\lambda) \le h_c^{ann}(\lambda)$ (this inequality on the critical curves is also referred to as annealed bound). It can be noticed also that the annealed free energy is not C^1 at criticality, that is the transition is of first order. Let us stress that the annealed free energy looses a lot of details of the original model (in particular: no trace of $K(\cdot)$!).

We have also $h_c(\lambda) > 0$ as soon as $\lambda > 0$, but this is not a trivial statement. In fact this means showing that $F(\lambda, 0) > 0$ for every $\lambda > 0$: in more dramatic terms, if the interaction does not select, on the average, a preferred solvent ($h = 0$), the polymer is localized even at arbitrarily weak coupling, a result established first in [26]. We skip the proof of this fact (in section 2 the proof of stronger results is sketched) and we simply observe that, together with the annealed upper bound, it implies that $h_c(\lambda) \longrightarrow 0$ as $\lambda \searrow 0$.

At this point convexity can be used in a very profitable way: since $\{(\lambda, y) : F(\lambda, y/\lambda) \le 0)\}$ is a convex set, its lower boundary $\lambda \mapsto \lambda h_c(\lambda)$ is a convex function. So we can write $h_c(\lambda) = g(\lambda)/\lambda$, with $g(\cdot)$ convex such that $g(\lambda) = o(\lambda)$ as $\lambda \searrow 0$. This directly implies in particular continuity of $h_c(\cdot)$ and, with a little bit more of work, also the fact that $h_c(\cdot)$ is strictly increasing [4].

Quite a bit of effort has been put into pinning down the value of $h_c(\cdot)$. Figure 2 sums up the results that are known on $h_c(\cdot)$ and, in particular, the content of

Theorem 1. *For every $\lambda > 0$ the following explicit bounds hold:*

$$\frac{1}{2\lambda/(1+\alpha)} \log M\left(-2\lambda/(1+\alpha)\right) \le h_c(\lambda) < \frac{1}{2\lambda} \log M\left(-2\lambda\right) = h_c^{ann}(\lambda), \quad (17)$$

where the left inequality is strict when $\alpha \ge 0.801$ (at least for λ small).

The lower bound in (17) is proved in [4]. The strict inequality in the upper bound in (17) was first proved in [30] to hold for large λ and then extended to every $\lambda > 0$ in [5]. In section 3 we give an alternative, more direct proof.

Highlighted in Figure 2 is the small λ behavior of $h_c(\lambda)$. In fact, as more extensively explained in section 4, for $\alpha \in (0,1)$ we have $h_c(\lambda) \overset{\lambda \searrow 0}{\sim} m_\alpha \lambda$, with $m_\alpha > 0$ depending only on α. The slope m_α is therefore a universal feature of the model: it does not depend on the details of the disorder sequence ω and of the underlying renewal τ. The proof of such a result goes through showing that for λ and h small the free energy of the copolymer is close to the free energy of a suitable continuum polymer model.
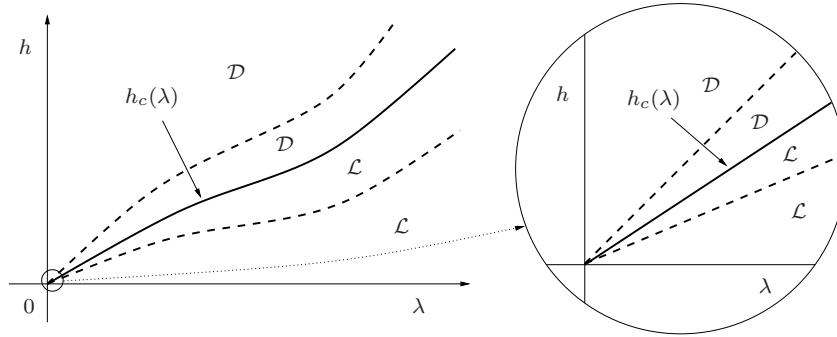
Fig. 2 On the left there is a sketch of the phase diagram and the critical curve $\lambda \mapsto h_c(\lambda)$. The localized (resp. delocalized) regime corresponds to (λ, h) lying strictly below (resp. above) the critical curve. Explicit upper and lower bound on $h_c(\cdot)$ are known, cf. Theorem 1, and they are schematically drawn as dashed lines. On the right one finds a zoom of the region near the origin, where the critical curve is close to a straight line and essentially all the relevant information close to the origin (weak coupling regime) is encoded in the slope of this line: while the details of the critical curve do depend on the law of the disorder ω and on the details of $K(\cdot)$, the slope depends only on α. This is an important universality feature to which section 4 is devoted.

It becomes therefore quite relevant to get a hold of the value of m_α (at least for $\alpha \in (0,1)$). As a matter of fact, from (17) one directly extracts

$$\frac{1}{1+\alpha} \leq m_\alpha \leq 1, \qquad \text{for every } \alpha > 0. \tag{18}$$

but this result can be sharpened to

$$\max\left(\frac{1}{2}, \frac{g(\alpha)}{\sqrt{1+\alpha}}, \frac{1}{1+\alpha}\right) \leq m_\alpha < 1, \tag{19}$$

where $g(\cdot)$ is a continuous function (of which we have an expression in terms of the primitive of an explicit function) such that $g(\alpha) = 1$ for $\alpha \geq 1$ and for which one can show that $g(\alpha)/\sqrt{1+\alpha} > 1/(1+\alpha)$ for $\alpha \geq 0.801$ (by evaluating $g(\cdot)$ numerically one can extend this result to $\alpha \geq 0.65$). Since the existence of m_α is not guaranteed for $\alpha \geq 1$, to be precise both in (18) and in (19) m_α should be replaced by the inferior and superior limits of $h_c(\lambda)/\lambda$.

Remark 4. For sake of conciseness we have left aside the $\alpha = 0$, that would require replacing the power law behavior (3) with a regularly varying behavior (allowing, in particular, the presence of logarithmic multiplicative corrections). The bounds that we have just presented directly generalize [15, Ch. 6] and for $\alpha = 0$ the three terms in (17) coincide.

We would like to stress that Theorem 1 and the bounds (19) show that some claims in the physical literature are wrong. Notably in [13, 32] it is claimed, for

$\alpha = 1/2$, that $h_c(\cdot)$ coincides with $h_c^{(ann)}(\cdot)$ (and (19) shows that also the weaker claim that $h_c(\lambda) \overset{\lambda \searrow 0}{\sim} h_c^{(ann)}(\lambda)$ is not correct. In [28, 23] it is claimed that, still for $\alpha = 1/2$, the inequality in the left-hand side of (17) is an equality. Theorem 1 falls short of proving that also this claim is false (even if it suggests it). A numerical study [9] lead for $\alpha = 1/2$, complemented by a careful statistical analysis using concentration inequalities, strongly suggests that the lower bound in (17) is strict and that the critical curve is somewhat *halfway* between the lower and the upper bound.

What the previous results and discussion expose is that a convincing heuristic theory predicting the location of the critical curve, or just its slope at the origin, is lacking. In this sense we consider that capturing the value of m_α, for $\alpha \in (0,1)$, is an important open problem. Since computing explicitly quenched quantities may be really out of reach, here are two sub-problems that are open:

⋆ show that $m_{1/2} > 2/3$;
⋆ is $m_\alpha = 1/(1+\alpha)$ for some $\alpha > 0$?

Finally, we point out that a *reduced (simplified) copolymer model* was introduced in [4], taking inspiration from the approach in [23]. The original hope was that this simplified model could catch the main features of the original copolymer model. However, this does not seem to be the case, since it has been shown [30, 6] that for the reduced model one has $m_\alpha = 1/(1+\alpha)$ for all $\alpha \in (0,1)$.

1.5 The critical behavior and a word about pinning models

Claims can be found in the physical literature about the critical behavior of this model (at least in the original set-up, $\alpha = 1/2$, *e.g.* [10, 21, 23, 32]), but these claims do not always agree with each other, apart for the fact that they all claim, not surprisingly, a smoothing effect of disorder. A rigorous result available on this issue has been proven in [19]: the transition of the general copolymer model is smooth (the derivative of the free energy vanishes at least linearly when the critical point is approached, hence it is at least Lipschitz continuous at the critical point), in contrast with the annealed case (where the the derivative of the free energy has a discontinuity at criticality, *cf.* Remark 3): for every $\lambda > 0$ there exists $c(\lambda) < \infty$ such that

$$\text{F}(\lambda, h_c(\lambda) + \delta) \leq (1+\alpha)c(\lambda)\delta^2 \qquad (20)$$

for every $\delta > 0$. The result was obtained under some technical conditions on the disorder law, which are satisfied for instance in the case of Gaussian or bounded charges. One can of course wonder whether the critical behavior of the copolymer model depends or not on α (and on λ?) and how, but once again, the substantial lack of sound physical predictions is quite disappointing. A natural open question however is:

⋆ Can one improve (20), in the sense of replacing the exponent 2 with a larger
  value?

A somewhat deeper insight into this issue can be achieved by considering also an-
other class of models, as we explain next.

The bound (20) in fact coincides with the one available for disordered pinning
models. Pinning models are a close companion to the copolymer, since the Boltz-
mann factor of a pinning model is

$$\exp\left(\sum_{n=1}^{N}(\beta\omega_n+h)\delta_n\right),\qquad(21)$$

where $\delta_n = \mathbf{1}_{n\in\tau}$, that is $\mathbf{1}_{S_n=0}$ in the random walk set-up (ω is chosen as before,
so that $\beta\omega_n+h$ is a random variable of mean $h\in\mathbb{R}$ and variance β^2). Therefore in
this case the polymer has an interaction with the environment only when it touches
the oil-water interface (or simply when it touches the x axis, usually called *defect
line*, since the model does not depend on the sign of the excursions). It is well
known that pining models exhibit a localization transition too and they can be dealt,
to a certain extent, with similar techniques [15]. However, in the end, there are
considerable differences, but let us try to single them out and see what they suggest
for copolymers:

1. The annealed pinning model is much richer than the annealed copolymer (*cf.* Re-
   mark 3). In particular, the annealed free energy does depend on $K(\cdot)$ and the crit-
   ical behavior depends on α: the transition is continuous (that is, the free energy
   is C^1) as soon as $\alpha \leq 1$ and it becomes smoother and smoother as α approaches
   0 [15]. Harris criterion (see references in [15, 16]) gives a precise prediction on
   what to expect for systems for which the annealed system has a transition that is
   sufficiently smooth (for the pinning case the criterion boils down to $\alpha < 1/2$):
   essentially it says that quenched and annealed systems have the same critical be-
   havior and it gives a precise prediction of the shift in the critical point due to
   the disorder (*irrelevant disorder regime*). At the same time it suggests/predicts
   that for $\alpha > 1/2$ *disorder is relevant*, even if arbitrarily weak. This scenario has
   now been made rigorous, see [16] and references therein, for pinning models.
   The crucial point for us is however the fact that the free energy of the annealed
   copolymer model is not differentiable at the critical point and therefore, in the
   Harris sense, disorder is always relevant.
2. Understanding critical phenomena when disorder is relevant is a major challenge
   and the possible scenarios set forth in the physical literature are quite intriguing,
   but very challenging and, at times, controversial (see *e.g.* [24, 33] and references
   therein). In this sense also the question that we have raised about improving (20)
   acquires particular importance.
3. When $\alpha > 1$, also the annealed pinning model free energy is not C^1, and the
   critical curve has been identified with no more precision than for the copolymer
   model. In fact the annealed pinning critical curve (again, a curve separating lo-
   calized and delocalized regimes, in the (β,h) plane) behaves like $-\beta^2/2$ when β

is small and the quenched critical curve is in $[-c_+\beta^2, -c_-\beta^2]$ for β small (with explicit values of the constants $0 < c_- < c_+ < 1/2$, cf. [11]). This is absolutely parallel to the fact that for the copolymer model $h_c(\lambda) \in [c_-\lambda, c_+\lambda]$, as one reads for example out of (18), (19).

Finally, it is natural to wonder what happens when a pinning interaction is added to the copolymer model, that is when not only the solvents are selective, but something special goes on at the interface (thus taking into account for example the lack of sharpness of the interface or the fact that impurities could be trapped at the interface). There are works on this model, often called *copolymer with adsorption* (see for example [25, 27, 34]), but the understanding is very limited: we refer to [15, § 6.3.2] for a detailed overview on this issue.

1.6 Organisation of the paper

The rest of the paper is devoted to going deeper into various results that we have stated, by giving either a sketch of arguments of proof, or alternative proofs and results that complement what can be found in the literature. More precisely:

- In section 2 we give a sketch of the proof on the lower bound in (17).
- In section 3 we give an alternative proof of the upper bound in (17).
- In section 4 we discuss the universality features of the copolymer model in the weak coupling regime, i.e., for small values of λ, h.
- Finally, section 5 is devoted to the description of the available results on the path properties of the copolymer model.

2 Localization estimates

The aim of this section is to give a sketch of the proof of the lower bound in Theorem 1, that is of the left inequality in (17), as well as of the left inequality in (19).

The key-phrase for the approaches in this section is: *rare stretch strategies*. The idea, inspired by the renormalization group approach in [23], is to restrict the partition function to polymer trajectories that can visit the lower half-plane only when there is a stretch of monomers that are *particularly, and anomalously, hydrophilic*. To do this we introduce an intermediate scale ℓ (large, but fixed) and look at the sequence of charges in blocks of ℓ charges at a time. We will consider two strategies (A and B) and, for simplicity, we will assume $\omega_1 \sim \mathcal{N}(0,1)$:

A: the jth block is good if $\sum_{n=(j-1)\ell+1}^{j\ell}(\omega_n + h) \leq -m\ell$, with m a positive value to be chosen below. For ℓ large, the probability that a given block is good is very small, about $\exp(-\ell(h+m)^2/2)$, so that good blocks are typically separated by a distance of about $\exp(\ell(h+m)^2/2)$.

B: the jth block is good if $\sum_{n=(j-1)\ell+1}^{j\ell}(\omega_n + h) = o(\ell)$. For ℓ large, the probability that a given block is good is again very small for $h > 0$, about $\exp(-\ell h^2/2)$, so that good blocks in this case are typically separated by a distance of about $\exp(\ell h^2/2)$.
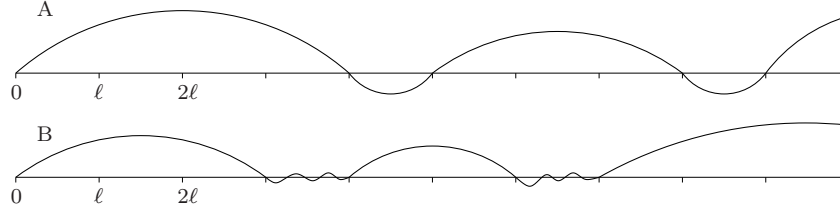
Fig. 3 The lower bound strategies presented in this section are two and they are both based on selecting some *good blocks*, just by looking at the environment. They are actually blocks in which the environment is atypically negative: in strategy A we really select blocks in which the empirical average of the charges $\omega_n + h$ is smaller that a value $-m < 0$ (selected in the end, in order to maximize the gain) and in strategy B we just aim at an empirical average close to zero (a rare event anyway, for $h > 0$). Then for strategy A we make a lower bound on the partition function by visiting the lower half plane if and only if a block is good and by insisting that in a good block the walk stays in the lower half plane. In strategy B we target again the good blocks, but we put no constraint on the walk in the good blocks.

Given a sequence ω of charges, the good blocks are identified by the rules just given and we introduce a set of polymer trajectories $\Omega_{N,\omega}^A$, resp. $\Omega_{N,\omega}^B$, for the strategy A, resp. B, defined as follows. $\Omega_{N,\omega}^A$ is the set of trajectories that stay in the upper half plane except in presence of good blocks, when they stay in the lower half plane, see the upper part of Figure 3. The set of polymer trajectories $\Omega_{N,\omega}^B$ still includes all trajectories that stay in the upper half plane except in presence of good blocks, but when there is a good block the only restriction is that the polymer has to touch the oil-water interface just before every good block and it has to touch it again at the end of the block, see the lower part of Figure 3.

The estimates are now just based on observing that

$$
Z_{N,\omega} \geq \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} \Delta_n(\omega_n + h)\right); \Omega_{N,\omega}^A\right]
$$

$$
= \frac{K(i_1\ell)}{2} \exp\left(-2\lambda \sum_{n=i_1\ell+1}^{(i_1+1)\ell}(\omega_n + h)\right) \frac{K(\ell)}{2} \times
$$

$$
\frac{K((i_2-i_1)\ell)}{2} \exp\left(-2\lambda \sum_{n=i_2\ell+1}^{(i_2+1)\ell}(\omega_n + h)\right) \frac{K(\ell)}{2} \times \ldots \quad (22)
$$

where the first good block is the i_1^{th} and so on. By using the definition of $\Omega_{N,\omega}^A$ we see that each term $-2\lambda \sum_{n=i_j\ell+1}^{(i_j+1)\ell}(\omega_n+h)$ is bounded below by $2\lambda m\ell$ and we notice that the right-hand side of (22) is a product of terms that, typically and to leading order, are just the same term, because the distance of good block is about $\exp(\ell(h+m)^2/2)$. Therefore this product of terms will give origin to an exponential growth in N (localization!) if

$$-\frac{1+\alpha}{2}\ell(h+m)^2+2\lambda\ell m+o(\ell)>0 \tag{23}$$

(we have of course used (3)). If we now optimize the choice of m we readily see that this condition is met if $h < \lambda/(1+\alpha)$, for ℓ sufficiently large. Therefore $h_c(\lambda) \geq \lambda/(1+\alpha)$, which is the lower bound in (17) for the special case of Gaussian charges (the extension to general disorder is straightforward).

Strategy B exploits a factorization similar to (22) but this time the contribution in the good blocks is of about $\exp(\text{F}(\lambda,0)\ell)$, since in the good blocks the empirical average of the charges is zero, so the charges are essentially centered and for large ℓ the random variables in a good block have a distribution that is close to centered IID standard Gaussian variables (in the complete proof, *cf.* [5, Sec. 2], this step is performed via a change of measure argument that makes rigorous the heuristics presented here: the argument is slightly more involved for non Gaussian charges). The analog of (23) in this case becomes

$$-\frac{1+\alpha}{2}\ell h^2+\text{F}(\lambda,0)\ell+o(\ell)\,. \tag{24}$$

At this point we need to estimate $\text{F}(\lambda,0)$. For example if one can show that $\text{F}(\lambda,0) \geq c\lambda^2$ for come $c>0$ (say, for $\lambda \leq \lambda_0$) then for the same values of λ one would have that $h_c(\lambda) \geq \lambda\sqrt{2c/(1+\alpha)}$. This is the basic idea leading to the (middle) lower bound in (19). Of course the work is now on estimating c. We will not go into this issue which, ultimately, is a refinement of the result in [26] and we refer to [5, Sec. 2] for details. But we point out that

- as explained in section 4, one can show that $\text{F}(\lambda,0)/\lambda^2$ has a positive limit that can be expressed in term of the free energy of a continuum polymer;
- in order to improve on the lower bound in (18) one needs $\sqrt{2c/(1+\alpha)} > 1/(1+\alpha)$, that is $c > 1/(2+2\alpha)$. This can be established, as recalled just below (19), for $\alpha \geq 0.65$. If we were to improve on the lower bound in (18) with this strategy for $\alpha = 1/2$ we would need to show $c > 1/3$, but numerical estimations suggest that $\lim_{\lambda\searrow 0}\text{F}(\lambda,0)/\lambda^2$ is smaller (probably by little) than $1/3$, so it is very likely that this strategy (barely) fails to establish that the lower bound in (18) can be made strict for $\alpha = 1/2$. The interest on this issue is because it would prove that the claims in [23, 28] are not correct.

3 Delocalization estimates

In this section we address the upper bound in Theorem 1, that is the right inequality in (17). We will actually present (in full) an argument that is substantially easier than the one originally used [5], even if it has the drawback to work only for $\alpha \in (0,1)$. This argument is still based on the fractional moment method (first used in the copolymer context in [30] to show that the upper bound in (17) holds for λ large and unbounded disorder), but it avoids the change of measure argument used in [5]. The change of measure argument in [5, 11] is an important technique, as well as its refinement in [31] that leads to the upper bound in (19), but we will not discuss these techniques here.

3.1 Fractional moment method: the general principle

We consider the fractional moment method in its most elementary application. For the constrained partition function (8) we can write

$$Z^{\mathsf{c}}_{N,\omega,\lambda,h} = \sum_{k=1}^{N} \sum_{0=t_0<t_1<...<t_k=N} \prod_{i=1}^{k} \frac{1+e^{-2\lambda(\Sigma_{t_{i-1}<j\le t_i}\omega_j+h(t_i-t_{i-1}))}}{2} K(t_i-t_{i-1}). \quad (25)$$

For $\gamma \in [0,1]$, from the inequality $(a+b)^\gamma \le a^\gamma + b^\gamma$, valid for all $a,b \ge 0$, we obtain the upper bound

$$\mathbb{E}((Z^{\mathsf{c}}_{N,\omega,\lambda,h})^\gamma) \le \sum_{k=1}^{N} \sum_{0=t_0<t_1<...<t_k=N} \prod_{i=1}^{k} \widetilde{K}_{\gamma,\lambda,h}(t_i-t_{i-1}), \quad (26)$$

where we define

$$\widetilde{K}_{\gamma,\lambda,h}(n) := \frac{1+e^{-(2\lambda\gamma h-\log\mathrm{M}(-2\lambda\gamma))n}}{2} K(n)^\gamma, \quad (27)$$

We also set

$$\Sigma(\gamma,\lambda,h) := \sum_{n\in\mathbb{N}} \widetilde{K}_{\gamma,\lambda,h}(n). \quad (28)$$

Assume that $\Sigma(\gamma,\lambda,h) \le 1$. By classical renewal theory [2], the right hand side of (26) equals the probability that a renewal process with step probability (or sub-probability, if $\Sigma(\gamma,\lambda,h) < 1$) $\widetilde{K}_{\gamma,\lambda,h}(\cdot)$ passes through N; in particular, it is bounded by 1. Then

$$\mathrm{F}(\lambda,h) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}(\log Z^{\mathsf{c}}_{N,\omega,\lambda,h}) = \lim_{N\to\infty} \frac{1}{\gamma N}\mathbb{E}(\log(Z^{\mathsf{c}}_{N,\omega,\lambda,h})^\gamma)$$

$$\le \lim_{N\to\infty} \frac{1}{\gamma N}\log\mathbb{E}((Z^{\mathsf{c}}_{N,\omega,\lambda,h})^\gamma) = 0,$$

whence $F(\lambda, h) = 0$ by (11).

This means that, when there exists $\gamma \in [0,1]$ such that $\Sigma(\gamma, \lambda, h) \leq 1$, it follows that $(\lambda, h) \in \mathscr{D}$. This allows to give explicit estimates on the delicalized region. Note that for $\gamma = 1$ we find the *annealed delocalized regime*, that we have already introduced: in fact $\Sigma(1, \lambda, h) \leq 1$ when $h > h_c^{ann}(\lambda) := \log M(-2\lambda)/(2\lambda)$. Since for $\gamma \in [0, 1/(1+\alpha)]$ one sees immediately that $\Sigma(\gamma, \lambda, h) = +\infty$, the interesting range is $\gamma \in (1/(1+\alpha), 1)$.

3.2 Fractional moment method: application

Let us define for $\lambda > 0$

$$\overline{h}(\lambda) := \inf\{h > 0 : \ \exists \gamma \in [0,1] \text{ such that } \Sigma(\gamma, \lambda, h) < 1\}. \tag{29}$$

Proposition 1. *For $\alpha \in (0,1)$ we have $h_c(\lambda) \leq \overline{h}(\lambda) < h_c^{ann}(\lambda)$ for every $\lambda > 0$.*

Proof. We have just remarked that $\Sigma(\gamma, \lambda, h) \leq 1$ implies $(\lambda, h) \in \mathscr{D}$, therefore $h_c(\lambda) \leq \overline{h}(\lambda)$. It remains to show that $\overline{h}(\lambda) < h_c^{ann}(\lambda)$ for every $\lambda > 0$, that is, for $\varepsilon > 0$ sufficiently small we can choose $\gamma \in [0,1]$ such that $\Sigma(\gamma, \lambda, h_c^{ann}(\lambda) - \varepsilon) < 1$. Note that for $\gamma \in (0,1)$

$$\frac{\partial \Sigma}{\partial \gamma}(\gamma, \lambda, h_c^{ann}(\lambda)) = \sum_{n \in \mathbb{N}} \frac{1 + e^{-(\gamma \log M(-2\lambda) - \log M(-2\lambda\gamma))n}}{2} (\log K(n)) K(n)^{\gamma}$$

$$- 2\lambda \left[(\log M)'(-2\lambda) + \frac{\log M(-2\lambda)}{2\lambda}\right] \sum_{n \in \mathbb{N}} \frac{e^{-(\gamma \log M(-2\lambda) - \log M(-2\lambda\gamma))n}}{2} n K(n)^{\gamma}.$$

By the strict convexity of $\log M(\cdot)$ and the fact that $\log M(0) = 0$,

$$(\log M)'(-2\lambda) < \frac{\log M(0) - \log M(-2\lambda)}{2\lambda} = -\frac{\log M(-2\lambda)}{2\lambda}. \tag{30}$$

Recalig our assumption (3), for $\alpha \in (0,1)$ we have $\sum_{n \in \mathbb{N}} n K(n) = \infty$, therefore by Fatou's lemma

$$\frac{\partial \Sigma}{\partial \gamma}(1^-, \lambda, h_c^{ann}(\lambda)) := \lim_{\gamma \uparrow 1} \frac{\partial \Sigma}{\partial \gamma}(\gamma, \lambda, h_c^{ann}(\lambda)) = +\infty. \tag{31}$$

Since $\Sigma(1, \lambda, h_c^{ann}(\lambda)) = 1$, it follows that $\Sigma(1 - \eta, \lambda, h_c^{ann}(\lambda)) < 1$, for $\eta > 0$ small enough. By continuity, $\Sigma(1 - \eta, \lambda, h_c^{ann}(\lambda) - \varepsilon) < 1$ for ε small enough, and the proof is completed. □

4 Continuum model and weak coupling limit

In this section we explain in some detail the universality feature sketched in Figure 2 and its caption. The idea is that at weak coupling the details of the model, that is the law of the renewal beyond the exponent α and the law of the disorder, are inessential and a suitable continuum model captures the leading behavior of the (large class of) discrete models we consider.

As we already remarked, it is convenient to look at the renewal process $\tau = \{\tau_k\}_{k \geq 0}$ as a random subset of $[0, \infty)$. It follows from our assumption (3) that the rescaled random set $\varepsilon\tau = \{\varepsilon\tau_k\}_{k \geq 0}$ converges in distribution as $\varepsilon \searrow 0$ toward a limit random set $\widetilde{\tau}^\alpha$, the so-called α-stable regenerative set (we refer to [12] for more details; cf. also [8]). This is a random closed subset of $[0, \infty)$ which is scale-invariant ($c\widetilde{\tau}^\alpha$ has the same law as $\widetilde{\tau}^\alpha$, for every $c > 0$), has zero Lebesgue measure and no isolated points. For $\alpha = \frac{1}{2}$ we have the representation $\widetilde{\tau}^\alpha = \{t \in [0, \infty) : B_t = 0\}$, where B is Brownian motion.

The complementary set $(\widetilde{\tau}^\alpha)^c$, being open, is the countable union of disjoint open intervals $\{I_n\}_{n \in \mathbb{N}}$. We can then define a continuous-time process $\widetilde{\Delta}^\alpha = \{\widetilde{\Delta}^\alpha_t\}_{t \in [0, \infty)}$, which is constant on each I_n and takes the value 0 or 1, decided by fair coin tossing: more precisely, in analogy with the discrete case, we set $\Delta_t := \sum_{n \in \mathbb{N}} \xi_n \mathbf{1}_{I_n}(t)$ where $\{\xi_n\}_{n \in \mathbb{N}}$ are i.i.d. $B(\frac{1}{2})$ random variables. For $\alpha = \frac{1}{2}$ we have the representation $\widetilde{\Delta}^\alpha_t = \mathbf{1}_{\{B_t < 0\}}$ with B a Brownian motion. In general, $\widetilde{\Delta}^\alpha$ may be viewed as the limit in distribution of the rescaled discrete process $\{\Delta_{\lfloor t/\varepsilon \rfloor}\}_{t \in [0, \infty)}$ as $\varepsilon \searrow 0$.

Let now $(\beta = \{\beta_t\}_{t \in [0, \infty)}, \mathbb{P})$ be a Brownian motion, independent of $(\widetilde{\Delta}^\alpha, \mathbf{P})$. We proceed for a moment in a somewhat informal way: as $a \searrow 0$, $a^{-1}\omega_{\lfloor t/a^2 \rfloor}$ converges toward the white noise $\mathrm{d}\beta_t / \mathrm{d}t$ and $\Delta_{\lfloor t/a^2 \rfloor}$ converges toward $\widetilde{\Delta}_t$, therefore

$$a\lambda \sum_{n=1}^{N/a^2} (\omega_n + ah)\Delta_n = \lambda \int_0^N (a^{-1}\omega_{\lfloor t/a^2 \rfloor} + h)\Delta_{\lfloor t/a^2 \rfloor}\,\mathrm{d}t \approx \lambda \int_0^N (\mathrm{d}\beta_t + h\,\mathrm{d}t)\widetilde{\Delta}^\alpha_t\,.$$

This hints at introducing a continuum partition function

$$\widetilde{Z}_{t,\beta} = \widetilde{Z}_{t,\beta,\lambda,h} := \mathbf{E}\left[\exp\left(-2\lambda \int_0^t \widetilde{\Delta}^\alpha_s(\mathrm{d}\beta_s + h\,\mathrm{d}s)\right)\right], \tag{32}$$

so that, recalling (5), one should have $Z_{N/a^2,\omega,a\lambda,ah} \approx \widetilde{Z}_{N,\beta,\lambda,h}$ for a small.

We now turn to precise statements. One can show that the definition (32) of the continuum partition function is well-posed, for $\mathbb{P}$-a.e. β, and one introduces the corresponding continuum free energy $\widetilde{\mathrm{F}}^\alpha(\lambda, h)$ in the usual way:

$$\widetilde{\mathrm{F}}_\alpha(\lambda, h) = \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \log \widetilde{Z}_{t,\beta,\lambda,h}\,. \tag{33}$$

The existence of this limit and the fact that it is self-averaging (i.e., the expectation $\mathbb{E}$ can be dropped) require a much longer and technical proof than the discrete counterpart, cf. [8].

Also the continuum free energy is non-negative: $\widetilde{F}_\alpha(\lambda, h) \geq 0$ for all $\lambda, h \geq 0$, as one can easily check. The localized and delocalized regimes can therefore be defined in analogy with the discrete case, namely $(\lambda, h) \in \widetilde{\mathscr{L}}$ (resp. $(\lambda, h) \in \widetilde{\mathscr{D}}$) if $\widetilde{F}_\alpha(\lambda, h) > 0$ (risp. $\widetilde{F}_\alpha(\lambda, h) = 0$), and it is easily shown that they are separated by a critical curve: $\widetilde{\mathscr{L}} = \{(\lambda, h) : h < \widetilde{h}_c^\alpha(\lambda)\}$ and $\widetilde{\mathscr{D}} = \{(\lambda, h) : h \geq \widetilde{h}_c^\alpha(\lambda)\}$. There is however a major simplification with respect to the discrete case: the scaling properties of the processes $\widetilde{\Delta}^\alpha$ and β yield easily $\widetilde{F}_\alpha(a\lambda, ah) = a^2 \widetilde{F}_\alpha(\lambda, h)$ for all $\lambda, h, a \geq 0$, therefore the critical curve is a straight line: $\widetilde{h}_c^\alpha(\lambda) = m_\alpha \lambda$ for some m_α.

We can finally come back to the rough consideration $Z_{N/a^2, \omega, a\lambda, ah} \approx \widetilde{Z}_{N, \beta, \lambda, h}$, that was discussed above. This can be made precise in the form of the following theorem.

Theorem 2. *For an arbitrary discrete α-copolymer model we have*

$$\lim_{a \searrow 0} \frac{1}{a^2} F(a\lambda, ah) = \widetilde{F}_\alpha(\lambda, h), \qquad \forall \lambda, h \geq 0. \tag{34}$$

Moreover

$$\lim_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} = m_\alpha. \tag{35}$$

This result was first proved in [7] in the special case of the basic model of section 1, i.e., for the discrete copolymer model based on the simple random walk on $\mathbb{Z}$, corresponding to $\alpha = \frac{1}{2}$ (in [17] one can find an argument to relax the assumption in [7] of binary charges and in [25] the case with *adsorption* is treated, *cf.* the end of section 1.5). The generalization to arbitrary α-copolymer models, with general disorder distribution, is in [8].

Note that (34) yields directly the existence of the limit as $\lambda \searrow 0$ of $F(\lambda, 0)/\lambda^2$, that was anticipated in section 2, as well as the fact that this limit coincides with $\widetilde{F}_\alpha(1, 0) > 0$. We also point out that (35) is not a direct consequence of (34).

The importance of Theorem 2 relies in its *universality* content: for any fixed $\alpha \in (0, 1)$ there is a single continuum model that captures the behavior of all discrete α-copolymer models for small values of λ and h. In other words, the differences among these models become irrelevant in the weak coupling limit. From this viewpoint, the slope m_α of the continuum critical curve is an extremely interesting object: improving the known bounds $\frac{1}{1+\alpha} \leq m_\alpha < 1$ would yield a substantial improvement in the understanding of the phase transition in this class of models.

5 Path properties

Up to now we have discussed the localization-delocalization transition only in terms of free energy. A complementary, and equally interesting, point of view is that of

looking at path properties. In other words, how does the typical (under $\mathbf{P}_{N,\omega}$, for typical ω) polymer trajectory look like? The bottom-line of the picture which has emerged up to this day is the following. In the localized region the polymer makes order of N excursions between the two half-planes; the lengths of such excursions are $O(1)$ and their distribution has an exponential tail. In the delocalized region, on the other hand, the number of monomers in the defavorable solvent (i.e. in the lower half plane) is not only sub-linear in N (this information can be obtained immediately from the fact that the free energy is zero there, cf. (12)) but it actually does not exceed $O(\log N)$, with high probability. In the following, we discuss this picture in a bit more detail.

5.1 The localized phase

This subsection is is extracted from [18], to which we refer for additional results, concerning for instance the exponential tail of the length of the polymer excursions between the two solvents. Path properties in the localized phase have been studied also in [1, 3].

Let $\mathbf{M}_N := \max_{i:\tau_i \leq N}(\tau_i - \tau_{i-1})$ be the length of the longest polymer excursion between the two solvents. The following result says that in the localized region correlations decay exponentially fast, and the longest excursion is of order $\log N$:

Theorem 3. *Let* $(\lambda,h) \in \mathscr{L}$. *There exist constants* c_1, c_2 *such that, for every pair of bounded local functions* A, B *of* τ *we have*

$$\sup_N \mathbb{E}\left[|\mathbf{E}_{N,\omega}(AB) - \mathbf{E}_{N,\omega}(A)\mathbf{E}_{N,\omega}(B)|\right] \leq c_1 \|A\|_\infty \|B\|_\infty e^{-c_2 d(A,B)}, \qquad (36)$$

where $d(A,B)$ *denotes the distance between the supports of A and B.*

Moreover, for every $\varepsilon \in (0,1)$ *the following holds in* $\mathbb{P}$-*probability:*

$$\lim_{N \to \infty} \mathbf{P}_{N,\omega}\left(\frac{1-\varepsilon}{\mu(\lambda,h)} \leq \frac{\mathbf{M}_N}{\log N} \leq \frac{1+\varepsilon}{\mu(\lambda,h)}\right) = 1, \qquad (37)$$

where μ *is defined as*

$$\mu(\lambda,h) = -\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\frac{1 + e^{-2\lambda \sum_{n=1}^{N}(\omega_n + h)}}{Z_{N,\omega}}. \qquad (38)$$

The existence of the limit (38), together with the bounds $0 < \mu(\lambda,h) \leq \mathrm{F}(\lambda,h)$ in the localized phase ($\mu(\lambda,h) \geq 0$ in general, and this is seen in the same way as for $\mathrm{F}(\lambda,h) \geq 0$), is proven in [18], where one can also find an argument showing that $\mu(\lambda,h) < \mathrm{F}(\lambda,h)$ under suitable (but not too restrictive) assumptions on the law of the charges.

As a simple consequence of the exponential decay of correlations one can prove that

1. the free energy is infinitely differentiable (in both λ and h) in the localized region $\mathscr{L}$;
2. for every bounded local observable A the limit $\lim_{N\to\infty} \mathbf{E}_{N,\omega}(A)$ exists $\mathbb{P}(\mathrm{d}\omega)$ almost surely and is reached exponentially fast.

As expected, the rate of exponential decay of the correlation functions (or inverse correlation length), i.e. c_2 in (36), tends to zero as $h \nearrow h_c(\lambda)$. In general, it is a very interesting open problem to understand the relation between this and the way the free energy vanishes close to the critical point. In [29], for the special case where $K(\cdot)$ is the law of the first return to zero of the symmetric simple random walk on $\mathbb{Z}$, it was proven that the best constant c_2 coincides with $\mu(\lambda,h)$ defined in (38).

There would be much to say about $\mu(\lambda,h)$ and $\mathrm{F}(\lambda,h)$ and we prefer to refer the reader to the introduction of [20] where this issue is treated in detail for pinning models. Here, in a simplistic way, we just point out that Theorem 3 and the bounds mentioned just after it are telling us in particular that $\mu(\lambda,h)$ is as good as $\mathrm{F}(\lambda,h)$ for detecting the localization transition. But:

⋆ is it true that $\log\mu(\lambda,h) \sim \log\mathrm{F}(\lambda,h)$ as $h \searrow h_c(\lambda)$? In view of the discussion on disorder relevance in section 1.5, we expect that this is not the case and establishing such a result would be very interesting.

5.2 The delocalized phase

Recall the definition $\mathscr{N}_N = \sum_{n=1}^{N} \Delta_n$ in (13). The following theorem shows that, strictly inside the delocalized region, $\mathscr{N}_N$ is typically at most of order $\log N$:

Theorem 4 ([17]). *For any $\delta > 0, \lambda > 0$ there exist $c > 0, q > 0$ such that for every $N \in \mathbb{N}$*

$$\mathbb{E}\, \mathbf{P}_{N,\omega,\lambda,h_c(\lambda)+\delta}(\mathscr{N}_N \geq n) \leq e^{-cn} \qquad \forall n \geq q\log N. \tag{39}$$

This result was proven in [17] under the assumption that the disorder law $\mathbb{P}$ satisfies a concentration inequality of sub-Gaussian type. This holds for instance in the case of Gaussian or bounded charges, and more generally whenever the distribution of ω_1 satisfies a Log-Sobolev inequality.

Here we give a simpler argument, inspired by [22], which works under the general assumptions of section 1 on the disorder law and gives the weaker statement

$$\mathbb{E}\, \mathbf{E}_{N,\omega,\lambda,h_c(\lambda)+\delta}(\mathscr{N}_N) \leq \frac{c}{2\lambda\delta}\log N \tag{40}$$

for some constant c. The same argument also shows that at the critical point $\mathscr{N}_N$ is typically at most of order $\sqrt{N\log N}$:

$$\mathbb{E}\, \mathbf{E}_{N,\omega,\lambda,h_c(\lambda)}(\mathscr{N}_N) \leq c'\sqrt{N\log N} \tag{41}$$

for some other constant c'.

Recalling (5), for all $h_0, h_1 \geq 0$ we can write

$$Z_{N,\omega,\lambda,h_0} := Z_{N,\omega,\lambda,h_1} \cdot \mathbf{E}_{N,\omega,\lambda,h_1} \left[\exp\left(2\lambda\left(h_1 - h_0\right)\mathscr{N}_N\right)\right]. \tag{42}$$

Restricting Z_{N,ω,λ,h_1} on the event $\{\tau_1 > N, \zeta_1 = 0\}$, we obtain the bound $Z_{N,\omega,\lambda,h_1} \geq \frac{1}{2}\sum_{n>N}K(n) \sim (const.)N^{-\alpha}$, by (3). Applying Jensen's inequality we obtain

$$\begin{aligned}
\frac{1}{N}\mathbb{E}\left[\log Z_{N,\omega,\lambda,h_0}\right] &\geq \frac{1}{N}\mathbb{E}\left[\log Z_{N,\omega,\lambda,h_1}\right] + \frac{2\lambda\left(h_1 - h_0\right)}{N}\mathbb{E}\,\mathbf{E}_{N,\omega,\lambda,h_1}\left[\mathscr{N}_N\right] \\
&\geq -c_1\frac{\log N}{N} + \frac{2\lambda\left(h_1 - h_0\right)}{N}\mathbb{E}\,\mathbf{E}_{N,\omega,\lambda,h_1}\left[\mathscr{N}_N\right],
\end{aligned} \tag{43}$$

where $c_1 \in (0, \infty)$. By (9) we have $\log Z^{\mathsf{c}}_{N,\omega,\lambda,h} \geq \log Z_{N,\omega,\lambda,h} - c_2 \log N$, for some constant c_2, therefore by (10) we can write, for some $c \in (0, \infty)$,

$$\mathrm{F}(\lambda, h_0) \geq \sup_{N \in \mathbb{N}}\left\{\frac{2\lambda\left(h_1 - h_0\right)}{N}\mathbb{E}\,\mathbf{E}_{N,\omega,\lambda,h_1}\left[\mathscr{N}_N\right] - c\frac{\log N}{N}\right\}. \tag{44}$$

- Take $h_0 = h_c(\lambda)$ and $h_1 = h_c(\lambda) + \delta$ with $\delta > 0$. Since $\mathrm{F}(\lambda, h_0) = 0$, from (44) we obtain (40).
- Now take $h_0 = h_c(\lambda) - \delta$, with $\delta > 0$, and $h_1 = h_c(\lambda)$. Since $\mathrm{F}(\lambda, h_c(\lambda) - \delta) \leq c_2\delta^2$ for some $c_2 = c_2(\lambda) \in (0, \infty)$ (cf. (20)), it follows again from (44) that

$$\limsup_{N\to\infty}\frac{\mathbb{E}\,\mathbf{E}^{\mathsf{c}}_{N,\omega,\lambda,h_c(\lambda)}\left[\mathscr{N}_N\right]}{\sqrt{N\log N}} \leq C, \qquad \text{where} \quad C := \frac{\sqrt{c_1\,c_2(\lambda)}}{\lambda} \in (0, \infty), \tag{45}$$

whence (41).

Remark 5. Recalling (27), let $\lambda, h > 0$ and $\gamma \in [0, 1]$ be chosen such that $\Sigma(\gamma, \lambda, h) < 1$, that is, $\widetilde{K}_{\gamma,\lambda,h}(\cdot)$ is a sub-probability kernel on $\mathbb{N}_0$. Since $\widetilde{K}_{\gamma,\lambda,h}(\cdot) \sim cK(N)^{\gamma} \sim c'N^{-\gamma(1+\alpha)}$ as $N \to \infty$ for some constants $c, c' > 0$, by (27) and (3), it is a basic result in renewal theory that the right hand side of (26) is asymptotically equivalent to $c''\widetilde{K}_{\gamma,\lambda,h}(N)$ as $N \to \infty$ for some constant $c'' > 0$, cf. [15, Theorem A.4]. Therefore we have for all $N \in \mathbb{N}$

$$\mathbb{E}\left((Z^{\mathsf{c}}_{N,\omega,\lambda,h})^{\gamma}\right) \leq C_1\widetilde{K}_{\gamma,\lambda,h}(N) \leq C_1 K(N)^{\gamma}, \tag{46}$$

where here and in the sequel C_i denotes a generic positive constant. Recalling (9) and (3), for the original (non constrained) partition function we have

$$\mathbb{E}\left((Z_{N,\omega,\lambda,h})^{\gamma}\right) \leq C_2 N^{\gamma} K(N)^{\gamma} \leq C_3 N^{-\gamma\alpha}. \tag{47}$$

This relation can be exploited to improve Theorem 4, showing that with high probability $\mathscr{N}_N$ is of order 1. More precisely, from the bound $Z_{N,\omega,\lambda,h} \geq \frac{1}{2}\sum_{n>N}K(n) \sim (const.)N^{-\alpha}$ and (42) it follows that for any $\delta > 0$

$$\mathbf{E}_{N,\omega,\lambda,h+\delta}\left[\exp\left(2\lambda\,\delta\,\mathscr{N}_N\right)\right] = \frac{Z_{N,\omega,\lambda,h}}{Z_{N,\omega,\lambda,h+\delta}} \leq C_4 N^\alpha Z_{N,\omega,\lambda,h}, \tag{48}$$

hence applying (47) we obtain

$$\mathbb{E}\left[\left(\mathbf{E}_{N,\omega,\lambda,h+\delta}\left[\exp\left(2\lambda\,\delta\,\mathscr{N}_N\right)\right]\right)^\gamma\right] \leq C_4^\gamma N^{\alpha\gamma} \mathbb{E}\left((Z_{N,\omega,\lambda,h})^\gamma\right) \leq C_5. \tag{49}$$

Recalling that $\gamma \in [0,1]$, by Markov's inequality we can write

$$\mathbb{E}\,\mathbf{P}_{N,\omega,\lambda,h+\delta}(\mathscr{N}_N \geq n) \leq \mathbb{E}\left(\mathbf{P}_{N,\omega,\lambda,h+\delta}(\mathscr{N}_N \geq n)^\gamma\right)$$
$$\leq e^{-2\lambda\delta\gamma n}\,\mathbb{E}\left[\left(\mathbf{E}_{N,\omega,\lambda,h+\delta}\left[\exp\left(2\lambda\,\delta\,\mathscr{N}_N\right)\right]\right)^\gamma\right] \leq C_5\,e^{-2\lambda\delta\gamma n}.$$

Summarizing, we have shown that whenever relation (47) holds true, there exists a constant $C > 0$ such that for every $\delta > 0$

$$\mathbb{E}\,\mathbf{P}_{N,\omega,\lambda,h+\delta}(\mathscr{N}_N \geq n) \leq C\,e^{-2\lambda\delta\gamma n}, \qquad \forall n \in \mathbb{N}. \tag{50}$$

We stress that relation (47) holds true in particular for every λ, h with $h > \bar{h}(\lambda)$ (with a suitable choice of $\gamma \in [0,1]$, recall (29) and Proposition 1), hence also *below* the annealed critical curve. This is therefore an improvement of equation (1.12) in [17].

We point out that delocalization properties were also studied in [3]. However the nature of the delocalized phase, in the pathwise sense, is still very little understood and, notably, almost sure results are laking. For example:

⋆ Is it true that, if (λ, h) is in the interior of $\mathscr{D}$, for every $\varepsilon > 0$ we have $\lim_{N\to\infty}\mathbf{P}_{N,\omega}(\Delta_n = 0 \text{ for every } n \in [\varepsilon N, N] \cap \mathbb{N}) = 0$ $\mathbb{P}(\mathrm{d}\omega)$-almost surely?

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